An electronic analogue for supersonic flow

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(*Received* 28 *January* 1957)

SUMMARY

An analogue method for solving certain quasi-linear hyperbolic partial differential equations is presented. The characteristic lines are formed by scanning electronically the screen of a cathode-ray tube. The boundary conditions are introduced in the form of an opaque mask. The solution appears as a picture on the screen of a second cathode-ray tube. The experiments demonstrate the feasibility of the approach, but the development of the machine has not been carried beyond this stage.

GENERAL DESCRIPTION OF THE METHOD

There are well-established analogue methods for solving Laplace's equation, such as the use of an electrolytic tank or of a soap film, to mention just two. An extensive bibliography of such methods has been given by Higgins (1956). On the other hand, there has not been comparable progress in finding analogue methods for solving the wave equation. Step-by-step numerical or graphical solutions are available. For example, for two-dimensional flows that are hyperbolic everywhere, the method of characteristics is applicable. It is essential to recapitulate certain features of the method of characteristics for the present point of view.

In the case of steady two-dimensional supersonic flow, there are two families of characteristic lines which can be regarded as a curvilinear coordinate system. In the case of a uniform supersonic flow, these characteristic lines become two sets of parallel straight lines each forming the Mach angle with the flow direction. If the departures from uniform flow are small, these characteristics will differ only little from the straight 'undisturbed' characteristics.

For the linearized theory of supersonic flow, the characteristics are assumed to have fixed direction (independent of the solution), and only the higher approximations involve the 'warping' of the characteristic lines.

Scanning methods are essential features of the television technique, and they can be quite readily adapted to generate the characteristic lines. A spot moving on the screen of a cathode-ray tube can sweep according to a scanning programme of arbitrary complexity. Present commercial television uses a scan pattern that consists of one family of parallel straight lines (usually in the horizontal direction from left to right) which are formed sequentially in time.

If a scanning programme is devised so that the ordinary television scan is performed twice, first tilted with a positive then with a negative angle, two families of fixed characteristics can be generated. For practical reasons it is more convenient to generate the two sets of lines as part of a single scan pattern as given in figure 1. By using symmetrical triangular waves for vertical deflection and saw-tooth waves for horizontal deflection with only slightly differing frequencies, one obtains a dense scan pattern. The scan lines also appear to be 'reflected' from the top and bottom edges of the scanned area.

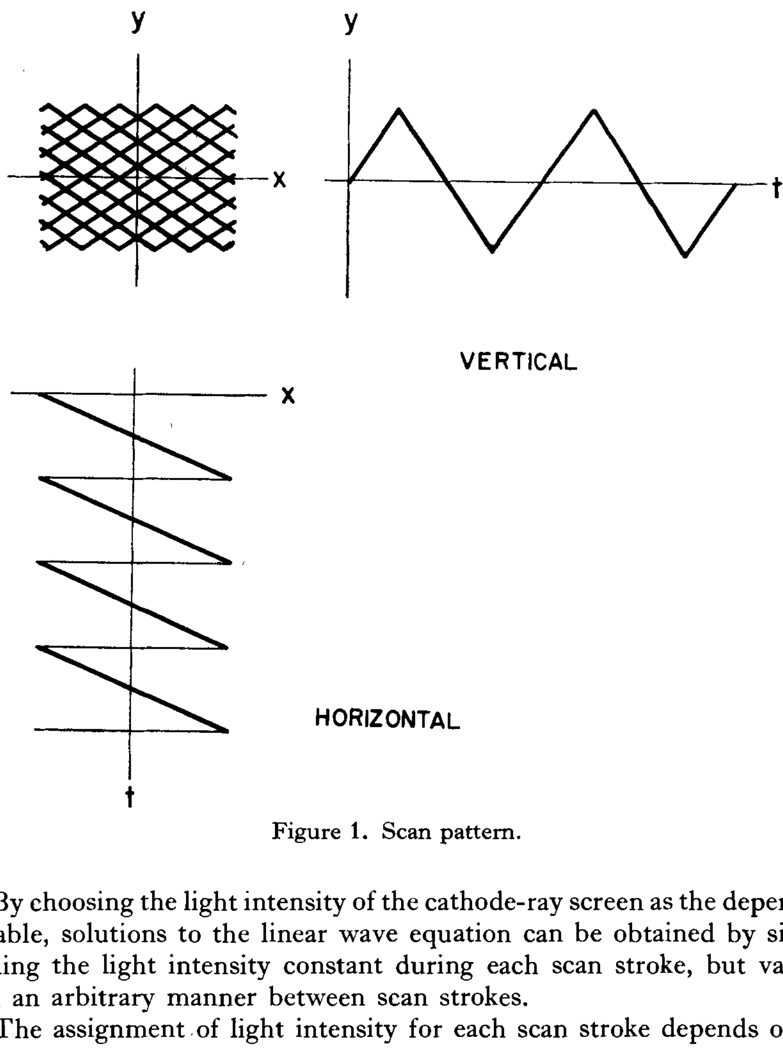


Figure 1. Scan pattern.

By choosing the light intensity of the cathode-ray screen as the dependent variable, solutions to the linear wave equation can be obtained by simply holding the light intensity constant during each scan stroke, but varying it in an arbitrary manner between scan strokes.

The assignment of light intensity for each scan stroke depends on the boundary conditions, and a simple method can be devised for introducing such boundary conditions. Suppose the light intensity is the integral of a function which is zero in the field, but not on the boundary. The spot will have constant intensity after it leaves the boundary and this constant can be set at the boundary. Pictures were obtained with the use of two cathode-ray tubes as indicated in figure 2. Identical scan patterns were applied to the two tubes. The first one, labelled *SCANNER*, sweeps in front of a cut-out mask giving light only when crossing the diamond-shaped slit. The light intensity is picked up by a phototube, and is integrated by an integrating circuit. The integrand is different from zero only when the scan spot passes the cut-out area. The value of the integral is displayed on the second tube labelled *DISPLAY*. The light intensity is constant along scan lines outside of the 'body'. Figure 3 (plate 1) shows the results of such an experiment.

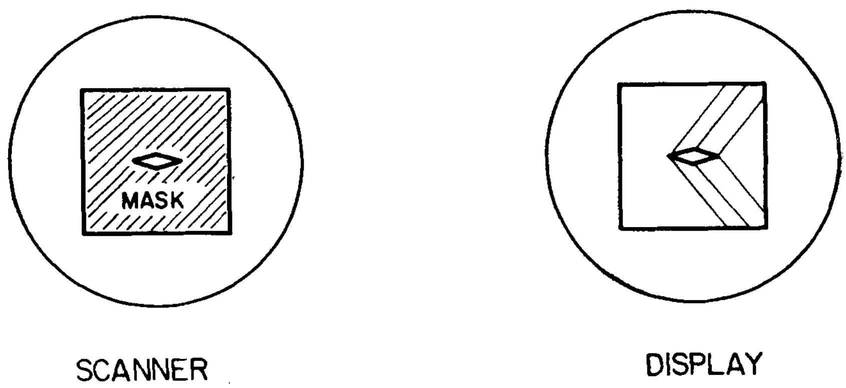


Figure 2. Two cathode-ray tubes.

The integrator can either be reset to zero at the extreme values of the vertical scan (figure 3 (*a*), plate 1), or permitted to hold its value and be reset only when returning to the left vertical edge (figure 3 (*b*), plate 1). This gives a different set of 'wall' boundary conditions.

In an actual flow problem, the boundary conditions are imposed by the slope of the surface and not by the integral of body thickness as in the above example. However, it is quite possible to construct masks so that the integrated values of light transmittance inside the body contour are proportional to the slopes of the contour.

The analogue for the linearized supersonic flow problem as described above appears to be quite trivial, and the interest in this method was mainly due to the fact that a certain amount of non-linear behaviour is rather easily introduced. If the disturbances are small but not infinitesimal, the principal effect of the non-linearity is to warp the characteristics by changing their local angle (and position) due to the change of the local Mach number and flow direction (see Lighthill 1954). In the analogue the simplest way to alter the characteristic angle is by changing one of the scan velocities. For example, the horizontal velocity of the cathode-ray spot can be altered by an amount proportional to the magnitude of the disturbance. Since the scan is obtained by integration, this change in velocity also modifies the position of the scan line.

For the non-linear operation, the boundary conditions are introduced in the same manner as for the linearized problem, namely, by scanning a mask as shown in figure 2. The video signal obtained from the phototube

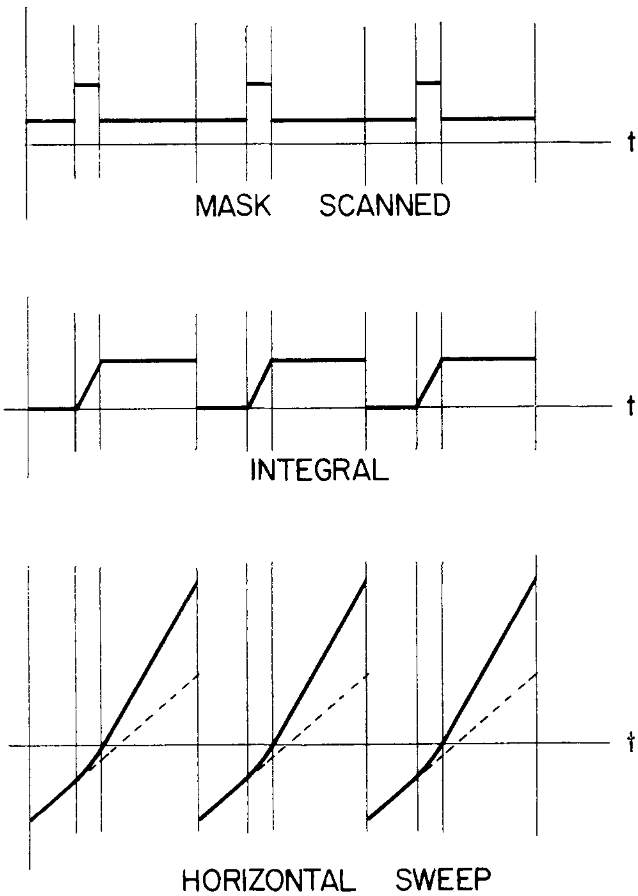


Figure 4. Formation of the horizontal sweep for non-linear operation.

by scanning the cut-out mask is integrated, and this integrated signal is considered as the 'disturbance' function. This signal can be used to modify either or both the horizontal scan velocity and the light intensity of the cathode-ray spot on the *DISPLAY* tube. Figure 4 shows the modifications of the horizontal scan by such an arrangement. Figure 5 shows a three-tone mask for a diamond aerofoil giving the correct boundary conditions when used with an integrator. Since the slope can be both positive and negative, the background is a neutral grey, and there are both positive (light) and negative (dark) areas contributing to the integral that represents the slope of the surface.

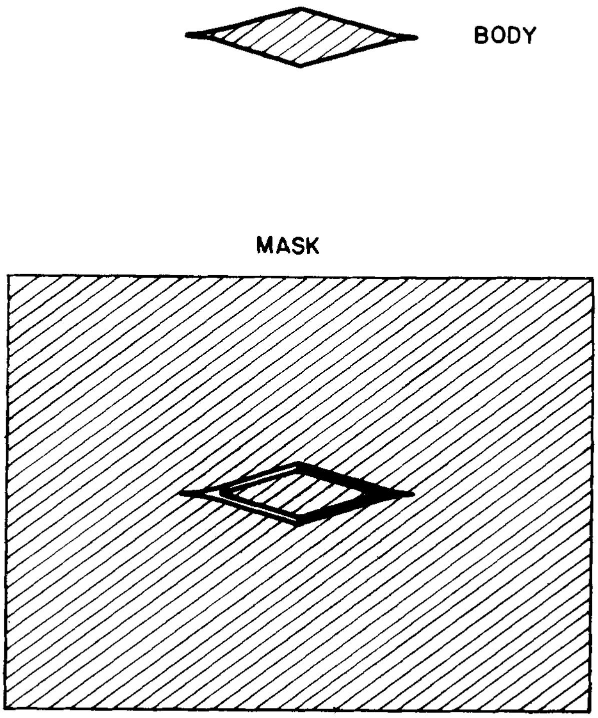


Figure 5. Proposed three-tone mask for introducing proper boundary conditions in case of diamond aerofoil.

Analysis

1. *The second-order equation*

The steady two-dimensional flow of a perfect gas has been the subject of extensive theoretical studies during the last hundred years. If the flow is assumed to be isentropic and the viscosity and heat conduction are neglected, the assumption of potential flow is justified.

Let us introduce the following quantities:

x, y, Cartesian coordinates,
$\phi(x,y)$, velocity potential,
$u(x,y)$, $v(x,y)$, velocity components in the x- and y-directions,
$q(x,y)$, absolute magnitude of the velocity,
c, local speed of sound,
q_∞, maximum possible velocity of the gas,
γ, ratio of the specific heats.

The local speed of sound depends only on the absolute magnitude of the local velocity,

$$c^2 = \tfrac{1}{2}(\gamma-1)(q_\infty^2 - q^2); \tag{1}$$

also

$$c_0 = \left(\frac{\gamma-1}{2}\right)^{1/2} q_\infty. \tag{1 a}$$

The governing equation of the flow is (see, for example, Howarth 1953)

$$(c^2-\phi_x^2)\phi_{xx} - 2\phi_x\phi_y\phi_{xy} + (c^2-\phi_y^2)\phi_{yy} = 0. \tag{2}$$

This second-order non-linear partial differential equation is designated as 'quasi-linear' because the highest-order derivatives $(\phi_{xx}, \phi_{xy}, \phi_{yy})$ all occur linearly, and only lower-order derivatives occur non-linearly.

The equation is elliptic where the flow is subsonic, i.e.

$$c^2 > q^2 = \phi_x^2 + \phi_y^2,$$

and hyperbolic where the flow is supersonic, i.e.

$$c^2 < q^2 = \phi_x^2 + \phi_y^2.$$

There are few exact solutions of this equation.

Let us introduce the concept of an undisturbed flow and a superimposed small perturbation

$$\phi(x,y) = U_1 x + c_1\psi(x,y); \tag{3}$$

then

$$u = U_1 + c_1\psi_x,$$

and

$$v = c_1\psi_y,$$

where $c = c_1$ when $\psi_x^2 + \psi_y^2 = 0$. It is also convenient to use $M_1 = U_1/c_1$, the Mach number of the undisturbed flow. From equation (1) we find that

$$c^2 = c_1^2[1 - \tfrac{1}{2}(\gamma-1)(2M_1\psi_x + \psi_x^2 + \psi_y^2)]. \tag{4}$$

By substituting c from (4) and ϕ from (3) into (2), we obtain a still exact equation for the disturbance potential ψ:

$$\begin{aligned}[(M_1^2-1) + \tfrac{1}{2}(\gamma+1)(2M_1+\psi_x)\psi_x + \tfrac{1}{2}(\gamma-1)\psi y^2]\psi_{xx} + {}& \\ + 2\psi_y(M_1+\psi_x)\psi_{xy} - {}& \\ - [1 - \tfrac{1}{2}(\gamma+1)\psi_y^2 - \tfrac{1}{2}(\gamma-1)(2M_1+\psi_x)\psi_x]\psi_{yy} = 0.& \quad (5)\end{aligned}$$

Equation (5) can be simplified by eliminating certain higher-order terms. Keeping only second-order terms and introducing $\beta_1^2 = M_1^2 - 1$, we obtain

$$[\beta_1^2 + (\gamma+1)M_1\psi_x]\psi_{xx} - [1-(\gamma-1)M_1\psi_x]\psi_{yy} = -2M_1\frac{\partial}{\partial x}(\psi_x^2). \tag{6}$$

Dividing through by $1-(\gamma-1)M_1\psi_x$, and again retaining only second-order terms in the expansion, we obtain

$$[\beta_1 + \kappa M_1\psi_x]^2\psi_{xx} - \psi_{yy} = -M_1\frac{\partial}{\partial x}(\psi_y^2), \tag{7}$$

where $\kappa = [(\gamma+1)+\beta_1^2(\gamma-1)]/2\beta_1$. Equation (7), which is exact up to the second order, is in a convenient form for our present purpose. The right-hand side is proportional to $v\,\partial v/\partial x$, and it vanishes as a second-order quantity when the x-axis is turned into the direction of the local flow. Since the purpose of the present analysis is to demonstrate the effect of non-linearity, the right-hand side of (7) is neglected and a modified second-order equation used:

$$[\beta_1+\kappa M_1\psi_x]^2\psi_{xx}-\psi_{yy} = 0. \tag{8}$$

If we introduce

$$\epsilon(x,y) = \kappa M_1\psi_x, \tag{9}$$

this becomes

$$(\beta_1+\epsilon)^2\epsilon_{xx}-\epsilon_{yy} = -2\beta_1\epsilon_x^2. \tag{10}$$

2. *The linearized equation*

The linearized equation for the present problem is the well-known wave equation

$$\beta_1^2\psi_{xx}-\psi_{yy} = 0. \tag{11}$$

The general solution is

$$\psi(x,y) = F(x+\beta_1 y)+G(x-\beta_1 y) \tag{12}$$

where F and G are arbitrary functions.

The pressure disturbance is computed in the following manner. For polytropic changes we have

$$p\rho^{-\gamma} = \text{const.}, \tag{13}$$

and the equation of state takes the form

$$\gamma p/\rho = c^2(q). \tag{14}$$

We see that all thermodynamic properties are unique functions of the absolute velocity q. The pressure takes the form

$$p = p_0(1-q^2/q_\infty^2)^{\gamma/(\gamma-1)}. \tag{15}$$

In the linearized approximation, we have

$$p = p_1-\tfrac{1}{2}\rho_1 U_1^2(2\psi_x/M_1) \tag{16}$$

$$\rho = \rho_1-\rho_1 M_1\psi_x. \tag{16 a}$$

If the wave equation is solved for a supersonic flow over a solid boundary $y=f(x)$, the boundary conditions are imposed in the form that the flow direction must follow the tangent of the solid boundary; thus

$$\frac{v}{u} = \frac{dy}{dx} = f' = \frac{\psi_y}{M_1+\psi_x} \doteqdot \frac{\psi_y}{M_1}. \tag{17}$$

If the oncoming flow is undisturbed, only one family of characteristics carries a signal ($G=0$, or $F=0$). One then obtains

$$\psi_y = \pm\,\beta_1\psi_x, \tag{18}$$

(the sign depends on whether F or G vanishes), and

$$p = p_1-\tfrac{1}{2}\rho_1 U_1^2(2f'/\sqrt{(M^2-1)}). \tag{19}$$

This shows that the local pressure disturbance is proportional to the local slope of the boundary.

The lines $\xi = x+\beta_1 y = \text{const.}$ and $\eta = x-\beta_1 y = \text{const.}$ are the characteristics, and they form a fixed network of oblique lines (Mach lines) along which the disturbances 'propagate'. The linearized theory improves as the disturbance becomes smaller (very slender body), and is really valid only for the limiting case of infinitesimal disturbances.

3. *The analogue system*

Two families of lines are generated, and they are regarded as two sets of characteristics. With the use of a finite number of scan lines, there will be only discrete values of ξ and η.

The coordinates of the cathode-ray spot during the nth stroke are given as a function of time by

$$X(t) = X_0+(t-t_0)\dot{X}, \tag{20}$$

$$Y(t) = (t-t_0)\dot{Y}. \tag{21}$$

In general $\dot{X}$ and $\dot{Y}$ are arbitrary functions of the time; X_0 and t_0 are arbitrary constants. By making a total of N strokes, the two sets of characteristics can be obtained in the following way.

Choose for the nth stroke

$$X_0 = \left.\begin{matrix} \xi_n & \text{for } n \text{ even,} \\ \eta_n & \text{for } n \text{ odd,} \end{matrix}\right\} \tag{22 a}$$

$$t_0 = n\,\Delta t, \tag{22 b}$$

where Δt is the period of one scan stroke, also

$$\dot{X} = \left.\begin{matrix} V(\beta_1+\epsilon_n) & \text{for } n \text{ even,} \\ V(\beta_1+\delta_n) & \text{for } n \text{ odd,} \end{matrix}\right\} \tag{22 c}$$

$$\dot{Y} = (-1)^n V. \tag{22 d}$$

Here ϵ_n and δ_n are numbers for $0 < n < N$, and they are introduced by the boundary conditions. For convenience, one can choose equal steps in ξ_n and η_n, i.e. $\xi_n = n\rho$, and $\eta_n = n\rho$. Using a dense set of lines, the subscript can be dropped, and ξ, η as well as δ, t can be considered as continuous variables. Thus

$$\xi = x-\beta_1 y-\epsilon(\xi)\, y, \tag{23 a}$$

$$\eta = x+\beta_1 y+\delta(\eta)\, y. \tag{23 b}$$

Observe that if $\epsilon = 0$ and $\delta = 0$, then $\xi = \text{const.}$, and $\eta = \text{const.}$ become the characteristics for the linearized problem.

If $\epsilon(x,y)$ and $\delta(x,y)$ are regarded as the dependent variables, it is easy to verify that they obey the first-order partial differential equations

$$(\beta_1+\epsilon)\epsilon_x+\epsilon_y = 0, \tag{24}$$

$$(\beta_1+\delta)\delta_x-\delta_y = 0. \tag{25}$$

The variables $\epsilon(x,y)$ and $\delta(x,y)$ each also obey the following second-order equation:

$$(\beta_1+\epsilon)^2\epsilon_{xx}-\epsilon_{yy} = -2\epsilon_x^2\beta_1-2\epsilon\epsilon_x^2, \tag{26}$$

$$(\beta_1+\delta)^2\delta_{xx}-\delta_{yy} = -2\delta_x^2\beta_1-2\delta\delta_x^2. \tag{27}$$

Naturally, these equations are valid separately, each for one family of simple waves only. By comparing (26) with (10), it is found that they agree for all except the last term on the right-hand side, but that is of the third order. This agreement can be interpreted in the following way.

The analogue generates simple wave solutions of the second-order hyperbolic partial differential equations (26) or (27). They are also simple wave solutions of the simplified second-order equation for the flow problem if a third-order term is disregarded. The dependent variable in both cases is the disturbance in the characteristic angle. For simple waves, the relation between ϵ and the flow angle θ is monotonic. This can be seen as follows.

Using only the linearized approach from (18), we find (taking the upper half-plane)

$$\psi_y = -\beta_1\psi_x. \tag{28}$$

The increment in the absolute magnitude of the velocity vector becomes

$$dq/q = -d\theta/\beta_1; \tag{29}$$

from (1), we get

$$dc/c = -\tfrac{1}{2}(\gamma-1)M_1^2\,dq/q; \tag{30}$$

and, from the definition of Mach number,

$$dM/M = dq/q-dc/c, \tag{31}$$

or

$$dM/M = -(1+\tfrac{1}{2}(\gamma-1)M_1^2)\,d\theta/\beta_1. \tag{32}$$

Also, the change in the cotangent of the characteristic angle with respect to flow direction becomes

$$d\beta = -M^2(1+\tfrac{1}{2}(\gamma-1)M_1^2)\,d\theta/\beta_1^2. \tag{33}$$

The characteristic makes an angle α with the x-axis given by

$$\cot\alpha = \beta_1+\epsilon, \tag{34}$$

$$\alpha = \mu+d\theta. \tag{35}$$

After linearization, this becomes

$$\beta_1+\epsilon = \beta_1+d\beta-M_1^2\,d\theta, \tag{36}$$

$$\epsilon = d\beta-M_1^2\,d\theta, \tag{37}$$

$$\epsilon = -\frac{M_1^4}{M_1^2-1}\frac{\gamma+1}{2}\,d\theta. \tag{38}$$

Equation (38) indicates that the disturbance to be imposed at the beginning of each scan stroke must be proportional to the surface slope, and the ratio constant is a function of the nominal Mach number of the flow.

Since the value of ϵ is constant during a scan stroke, the simplest way of imposing its value at the boundary is by integration. An integrating circuit holds the value of ϵ during the scan. It is reset after each stroke.

The signal is built up before the cathode-ray spot leaves the body contour The boundary condition can be introduced as a variable opacity or variable thickness of a zone within the body.

Figure 5 shows one simple method for introducing the right boundary condition.

4. *Light intensity of the screen*

The solution is obtained in the form of a photograph taken from the screen of the display cathode-ray tube. The light intensity on the cathode-ray tube is however, not proportional to $\epsilon(\xi)$, but depends on both the luminosity of the spot and on the duration the spot spends in the local area.

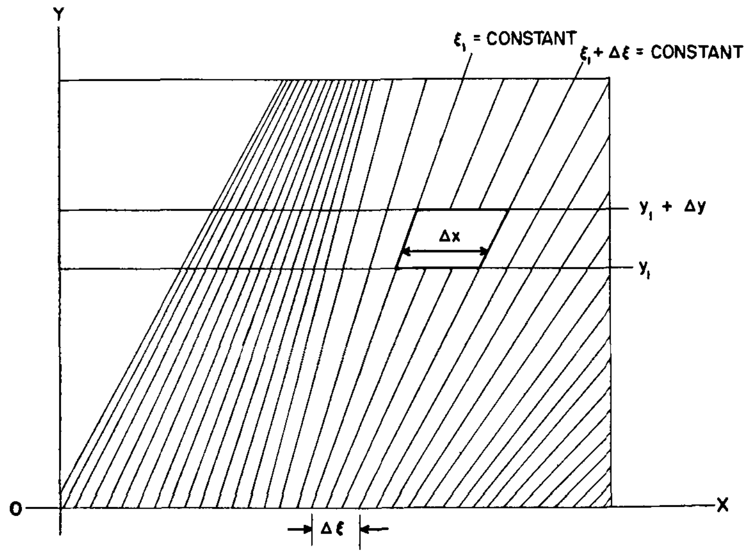


Figure 6. Light intensity relations.

In order to find the relations between ϵ and the screen intensity, introduce $i(t)$, the instantaneous luminous intensity of the cathode-ray spot, and $I(x,y)$, the average luminous intensity per unit area of the screen. The intensity is held constant during a single stroke but varies from stroke to stroke. Note that $i(t)$ and $I(x,y)$ do not have the same dimensions. Introduce for convenience

$$I_0(\xi) = \frac{i(t)T}{ab}, \tag{39}$$

where ab is the total scanned area and T is the time of the entire scan cycle. The local intensity obeys the equation of conservation of (luminous) energy

$$I(x,y)\,\Delta x\,\Delta y = i(t)\sum\Delta T, \tag{40}$$

where $\sum\Delta T$ is the sum of time intervals the spot spends in the area $\Delta x\,\Delta y$. The luminous intensity $I(x,y)$ can be expressed as shown in figure 6. The small quadrangle is bounded by $\xi_1 = \text{const.}$, $\xi_1+\Delta\xi = \text{const.}$, $y_1 = \text{const.}$, and $y_1 + \Delta y = \text{const.}$ Let us assume successive scan lines correspond to equal increments in ξ (they are equidistant on the x-axis). The fraction of the number of scan lines that pass through the area is $\Delta\xi/2a$ (the factor 2 is due to the fact that there are two families of scan lines during a full scan cycle). The fraction of the time the spot spends within the area during a single stroke is $\Delta y/b$ (the vertical velocity is constant), so that

$$\frac{\sum \Delta T}{T} = \frac{\Delta\xi\,\Delta y}{2ab}.$$

Substituting this value in (40), we obtain

$$2I_0(\xi)\,\Delta\xi = I(x,y)\,\Delta x,$$

or

$$I(x,y) = \tfrac{1}{2}I_0(\xi)\frac{\partial\xi}{\partial x}. \tag{41}$$

If the light intensity is modulated in intensity so that

$$I_0(\xi) \doteqdot \frac{d\epsilon}{d\xi}, \tag{42}$$

the luminous intensity of the screen becomes

$$I(x,y) \doteqdot \frac{d\epsilon}{d\xi}\frac{\partial\xi}{\partial x} = \frac{\partial\epsilon}{\partial x}. \tag{43}$$

This is a significant result since it was previously shown that ϵ is analogous to pressure (or density) within the range of linear approximation. In wind-tunnel experiments, the x derivative of the density is made visible optically by the schlieren method. This is the most common visualization method used in routine wind-tunnel work. If the luminosity of the spot (intensity modulation, or 'Z-axis') is varied according to the change of ϵ from scan stroke to scan stroke, schlieren pictures result.

Experiments

The experiments were carried out only to indicate feasibility of the method, and were not pursued to any degree of perfection. A flying-spot scanner previously described by Kovasznay & Joseph (1955) was modified for the present purpose, and many of the limitations of the original equipment determined the experiments that could be attempted. The block diagram of the equipment is shown in figure 7.

The flying-spot generator produces a video signal corresponding to the transparent and opaque areas of the mask. A real image is formed in the plane of the mask by an objective lens and the signal obtained at the phototube is of the 'yes or no' type. A bi-stable trigger quantizes the phototube signal into a definite two-level signal (0 or 1). The reset integrator integrates the signal during one scan stroke. The integrator is reset to zero either at the end of the horizontal strokes or only at the end of both vertical and horizontal strokes. In the former case, the waves 'reflect' from the vertical edges of the scan pattern (see figure 3, plate 1); in the latter they do not. If the masks were cut in such a way that the transparent portion along the scan line corresponded to the slope of the boundary, the signal obtained after integration becomes proportional to the ϵ disturbance. In the analogue for the linearized problem, we can simply modulate the light intensity with this signal. Figure 3 (plate 1) was obtained in such a way (with the switch, shown in figure 7, set on *INTENSITY CONTROL*). For non-linear operation, however, the horizontal scan velocity must be modified according to ϵ. This is obtained by the variable velocity sweep. This circuit is essentially a second reset integrator. In absence of a velocity control signal, it integrates a constant and $\dot{X} = \beta_1 V$, and it is reset at equal intervals. The velocity control signal is added to the integrand, forming $\dot{X} = V\beta_1 + V\epsilon$ as required by (22 c).

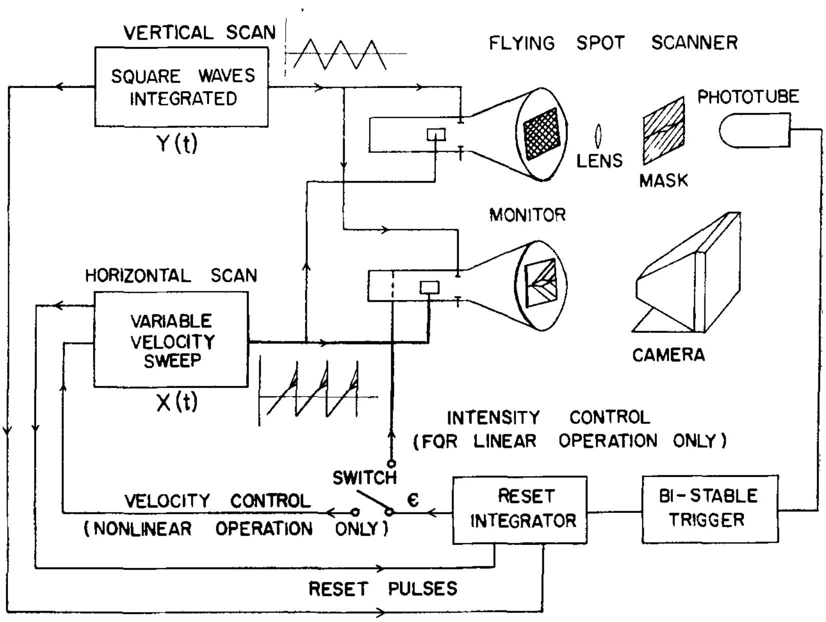
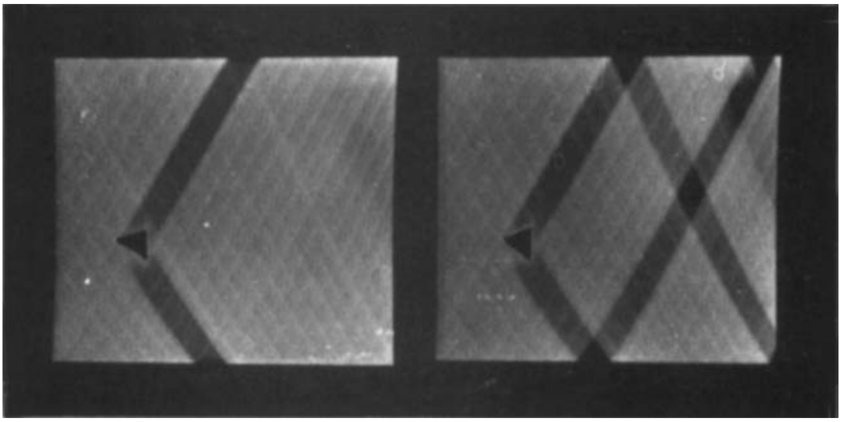


Figure 7. Block diagram of equipment.

In the present experiments, the light intensity I_0 was not varied according to (42), but was kept constant. In this way the flow pictures do not actually correspond to schlieren pictures.

A more serious defect of the present experimental set-up is that the boundary conditions were not introduced correctly. The masks were simply cut out of black paper, thus giving two levels of the video signal. Every smooth closed body contour, however, has both positive and negative slopes. In order to form both positive and negative disturbances, one needs at least a three-level discriminator instead of the bi-stable trigger, and this



(*a*) (*b*)

Figure 3. Linear operation. (*a*) Integrator reset both at the end of each vertical and horizontal scan stroke. (*b*) Integrator reset only at the end of horizontal scan stroke.

(*a*)

(*b*)

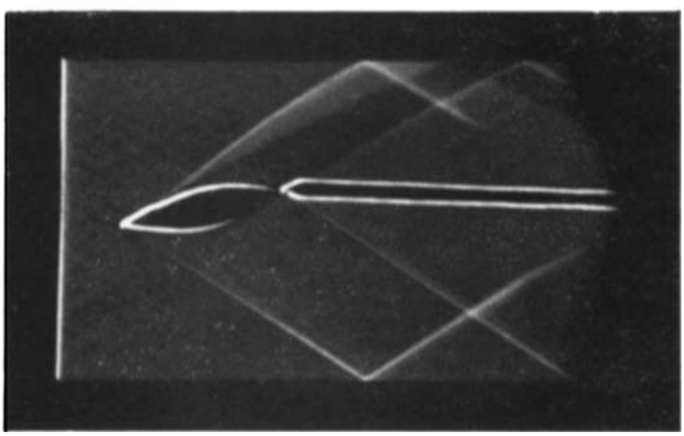


Figure 9. Non-linear analogue for supersonic flow. (*a*) Mask scanned. (*b*) Picture obtained on display.

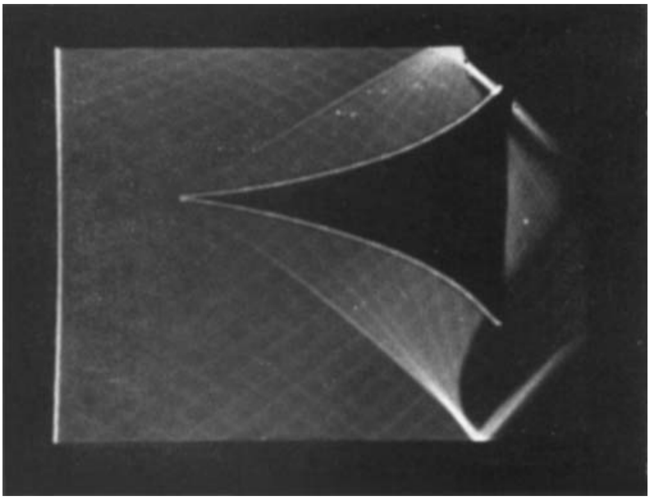
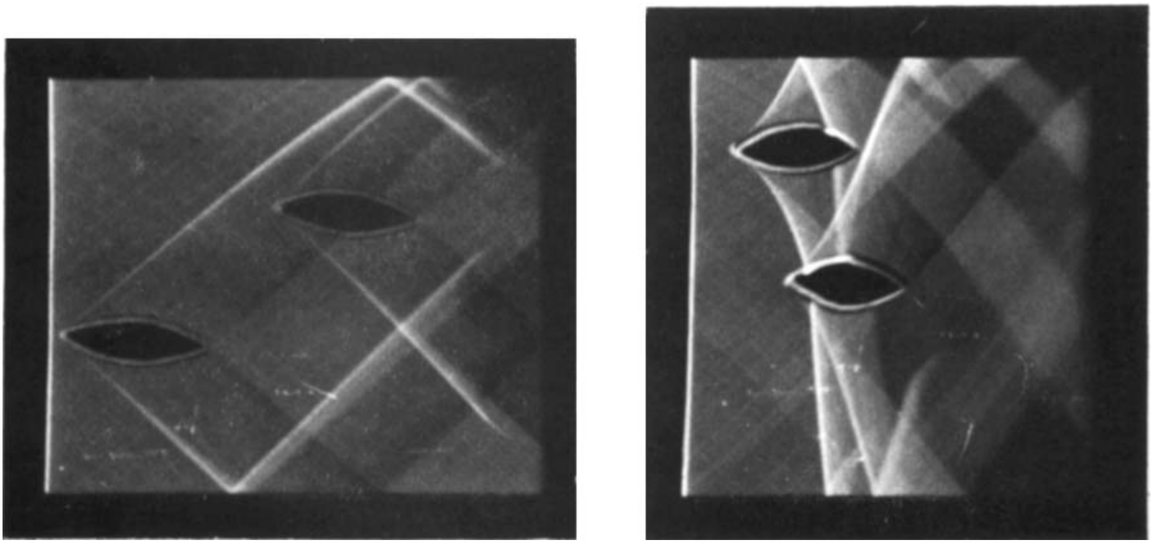


Figure 8. Crowding of characteristics over concave surface.



(a) (b)

Figure 10. Different degrees of non-linear effect. (a) Weak non-linear effect. (b) Strong non-linear effect.

was not available at the time of the experiments. There is really one exception to this comment. Figure 8 (plate 2) represents a concave 'body', and the slope is approximately proportional to the local thickness so in this particular example the proper boundary conditions were applied. (A wedge with exponential contour satisfies the requirement that thickness is proportional to slope.)

To modulate the light intensity in accordance with (42) would have required the use of a memory circuit to retain the value of ϵ from the previous scan stroke, hence to form $d\epsilon/d\xi \doteqdot \epsilon(\xi_n) - \epsilon(\xi_{n-1})$.

It should be emphasized that both of these improvements (proper boundary conditions by use of three-level discriminator and light-intensity modulation according to (42)) require only standard electronic design, and were not attempted only for lack of time available for the work.

Figure 8 (plate 2) clearly shows the crowding of characteristics into the formation of an envelope. Figure 9 (plate 1) shows the photographic print of the cut-out mask and the 'flow picture' thus obtained. Since the period of scan cycle is constant and the horizontal scan velocity is variable, the right edge of the picture is irregular as the total x distance is varying with the disturbance. In order to render the pictures more 'life-like', a small artifice was introduced. On the display cathode-ray tube intensity grid, a linear combination of the video signal and its time derivative was imposed. The negative value of the video signal from the phototube produced darkness in the interior of the body and the time derivative of the video signal gave a sharp outline of the body contours. Without this artifice the body contours would not even have been visible.

Figure 10 (plate 2) shows two objects. In the picture in figure 10(a) the non-linearity is 'weak', having low amplification of the disturbance signal. In the other picture, the non-linear effect is strong.

As a possible extension of the method, the analogue for axisymmetric supersonic flows seems to be quite feasible. A more speculative proposition is the use of iteration. Using iterative processes, the first flow picture would be obtained by scanning a cut-out mask representing the boundary conditions. The display would be photographed and would be introduced into the scanner to obtain a second approximation. In this way, the interaction between the two families of characteristics would be introduced.

The extension to the one-dimensional unsteady problem, where the cathode-ray face would represent the x, t plane, is also obvious. It is interesting to note that in this case the analogue becomes exact for $\gamma = 3$, as shown in the Appendix.

The present work was carried out in the National Bureau of Standards, Washington, D.C., under a cooperative programme of basic instrumental research and development while the author was a consultant. The author acknowledges the enthusiastic help of Mr H. M. Joseph and Mr N. Newman. Thanks are due to Dr B. T. Chu of Brown University for contributing the special case given in the Appendix.

Appendix

The one-dimensional unsteady flow of a compressible gas presented on an (x, t)-plane has a qualitative similarity to the steady two-dimensional flow. The governing equations for an isentropic perfect gas are:

Conservation of mass, $$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0. \tag{A 1}$$

Conservation of momentum, $$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{c^2}{\rho}\frac{\partial \rho}{\partial x} = 0. \tag{A 2}$$

By eliminating ρ, one obtains the pair of equations

$$\frac{\partial}{\partial t}\left(u + \frac{2}{\gamma - 1}c\right) + (u+c)\frac{\partial}{\partial x}\left(u + \frac{2}{\gamma-1}c\right) = 0, \tag{A 3}$$

$$\frac{\partial}{\partial t}\left(u - \frac{2}{\gamma - 1}c\right) + (u-c)\frac{\partial}{\partial x}\left(u - \frac{2}{\gamma-1}c\right) = 0. \tag{A 4}$$

Introduce the characteristic lines $\xi = \text{const.}$ and $\eta = \text{const.}$, i.e. with $\xi = \xi(x, t)$, $\eta = \eta(x, t)$, so that

$$d\xi = dx - (u+c)\,dt, \qquad d\eta = dx - (u-c)\,dt. \tag{A 5}$$

Along these lines the characteristic relations are obtained:

$$\text{Along} \quad \xi = \text{const.}, \qquad u + 2c/(\gamma - 1) = \text{const.}$$

$$\text{Along} \quad \eta = \text{const.}, \qquad u - 2c/(\gamma - 1) = \text{const.}$$

The characteristic relations can be conveniently written as

$$c + \tfrac{1}{2}(\gamma - 1)u = c_0 + \epsilon(\xi), \qquad c - \tfrac{1}{2}(\gamma-1)u = c_0 + \delta(\eta), \tag{A 6}$$

where $\epsilon(\xi)$ and $\delta(\eta)$ are arbitrary functions of their arguments. In general the characteristics will be curved lines. However, if $\gamma = 3$,

$$u + c = c_0 + \epsilon(\xi), \qquad u - c = -c_0 - \delta(\eta). \tag{A 7}$$

Consequently, the equations (A 5) can be independently integrated, giving, for instance,

$$\xi = x - [c_0 + \epsilon(\xi)]t, \qquad \eta = y + [c_0 + \delta(\eta)]t. \tag{A 8}$$

These equations represent two 'non-interacting' sets of characteristics.

Unfortunately all gases have $\gamma < 5/3$. It is worth mentioning that shallow water flow has properties analogous to a two-dimensional gas flow with $\gamma = 2$ (see Courant & Friedrichs 1948).

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